ANDÔ DILATIONS AND INEQUALITIES ON NONCOMMUTATIVE VARIETIES

GELU POPESCU

Abstract. Andô proved a dilation result that implies his celebrated inequality which says that if T_1 and T_2 are commuting contractions on a Hilbert space, then for any polynomial p in two variables,

$$\|p(T_1, T_2)\| \leq \|p\|_{\mathbb{D}^2},$$

where $\mathbb{D}^2$ is the bidisk in $\mathbb{C}^2$. We find analogues of Andô's results for the elements of the bi-ball $\mathbf{P}_{n_1,n_2}$ which consists of all pairs $(\mathbf{X}, \mathbf{Y})$ of row contractions $\mathbf{X} := (X_1, \ldots, X_{n_1})$ and $\mathbf{Y} := (Y_1, \ldots, Y_{n_2})$ which commute, i.e. each entry of $\mathbf{X}$ commutes with each entry of $\mathbf{Y}$. The results are presented in a more general setting, namely, when $\mathbf{X}$ and $\mathbf{Y}$ belong to noncommutative varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ determined by row contractions subject to constraints such as

$$q(X_1, \ldots, X_{n_1}) = 0 \quad \text{and} \quad r(Y_1, \ldots, Y_{n_2}) = 0, \qquad q \in \mathcal{P}, r \in \mathcal{R},$$

respectively, where $\mathcal{P}$ and $\mathcal{R}$ are sets of noncommutative polynomials. We obtain dilation results which simultaneously generalize Sz.-Nagy dilation theorem for contractions, Andô's dilation theorem for commuting contractions, Sz.-Nagy–Foiaş commutant lifting theorem, and Schur's representation for the unit ball of H^∞, in the framework of noncommutative varieties and Poisson kernels on Fock spaces. This leads to an Andô type inequality on noncommutative varieties, which, in the particular case when $n_1 = n_2 = 1$ and T_1 and T_2 are commuting contractive matrices with spectrum in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, takes the form

$$\|p(T_1, T_2)\| \leq \min\left\{\|p(B_1 \otimes I_{\mathbb{C}^{d_1}}, \varphi_1(B_1))\|, \|p(\varphi_2(B_2), B_2 \otimes I_{\mathbb{C}^{d_2}})\|\right\},$$

where $(B_1 \otimes I_{\mathbb{C}^{d_1}}, \varphi_1(B_1))$ and $(\varphi_2(B_2), B_2 \otimes I_{\mathbb{C}^{d_2}})$ are analytic dilations of (T_1, T_2) while B_1 and B_2 are the universal models associated with T_1 and T_2, respectively. In this setting, the inequality is sharper than Andô's inequality and Agler-McCarthy's inequality.

We show that there is a universal model $(S \otimes I_{\ell^2}, \varphi(S))$, where S is the unilateral shift and $\varphi(S)$ is an isometric analytic Toeplitz operator on $H^2(\mathbb{D}) \otimes \ell^2$, such that

$$\|[p_{rs}(T_1, T_2)]_k\| \leq \|[p_{rs}(S \otimes I_{\ell^2}, \varphi(S))]_k\|,$$

for any commuting contractions T_1 and T_2 on Hilbert spaces, any $k \times k$ matrix $[p_{rs}]_k$ of polynomials in $\mathbb{C}[z, w]$, and any $k \in \mathbb{N}$. Analogues of this result for the bi-ball $\mathbf{P}_{n_1,n_2}$ and for a class of noncommutative varieties are also considered.