ANDŐ DILATIONS AND INEQUALITIES ON NONCOMMUTATIVE VARIETIES

GELU POPESCU

Abstract. Andő proved a dilation result that implies his celebrated inequality which says that if $T_1$ and $T_2$ are commuting contractions on a Hilbert space, then for any polynomial $p$ in two variables,

$$\|p(T_1, T_2)\| \leq \|p\|_{D^2},$$

where $D^2$ is the bidisk in $\mathbb{C}^2$. We find analogues of Andő’s results for the elements of the bi-ball $P_{n_1, n_2}$ which consists of all pairs $(X, Y)$ of row contractions $X := (X_1, \ldots, X_{n_1})$ and $Y := (Y_1, \ldots, Y_{n_2})$ which commute, i.e. each entry of $X$ commutes with each entry of $Y$. The results are presented in a more general setting, namely, when $X$ and $Y$ belong to noncommutative varieties $V_1$ and $V_2$ determined by row contractions subject to constraints such as $q(X_1, \ldots, X_{n_1}) = 0$ and $r(Y_1, \ldots, Y_{n_2}) = 0$, $q \in \mathcal{P}$, $r \in \mathcal{R}$, respectively, where $\mathcal{P}$ and $\mathcal{R}$ are sets of noncommutative polynomials. We obtain dilation results which simultaneously generalize Sz.-Nagy dilation theorem for contractions, Andő’s dilation theorem for commuting contractions, Sz.-Nagy–Foiaş commutant lifting theorem, and Schur’s representation for the unit ball of $H^\infty$, in the framework of noncommutative varieties and Poisson kernels on Fock spaces. This leads to an Andő type inequality on noncommutative varieties, which, in the particular case when $n_1 = n_2 = 1$ and $T_1$ and $T_2$ are commuting contractive matrices with spectrum in the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$, takes the form

$$\|p(T_1, T_2)\| \leq \min \{\|p(B_1 \otimes I_{C^{n_1}}, \varphi_1(B_1))\|, \|p(\varphi_2(B_2), B_2 \otimes I_{C^{n_2}})\|\},$$

where $(B_1 \otimes I_{C^{n_1}}, \varphi_1(B_1))$ and $(\varphi_2(B_2), B_2 \otimes I_{C^{n_2}})$ are analytic dilations of $(T_1, T_2)$ while $B_1$ and $B_2$ are the universal models associated with $T_1$ and $T_2$, respectively. In this setting, the inequality is sharper than Andő’s inequality and Agler-McCarthy’s inequality.

We show that there is a universal model $(S \otimes I_{\ell^2}, \varphi(S))$, where $S$ is the unilateral shift and $\varphi(S)$ is an isometric analytic Toeplitz operator on $H^2(D) \otimes \ell^2$, such that

$$\|p_{rs}(T_1, T_2)\|_k \leq \|p_{rs}(S \otimes I_{\ell^2}, \varphi(S))\|_k,$$

for any commuting contractions $T_1$ and $T_2$ on Hilbert spaces, any $k \times k$ matrix $[p_{rs}]_k$ of polynomials in $\mathbb{C}[z, w]$, and any $k \in \mathbb{N}$. Analogues of this result for the bi-ball $P_{n_1, n_2}$ and for a class of noncommutative varieties are also considered.